arXiv:1211.3335v1 [math.FA] 14 Nov 2012

THE FIXED POINT PROPERTY IN A BANACH SPACE ISOMORPHIC TO c_0

COSTAS POULIOS

Abstract. We consider a Banach space, which comes naturally from c_0 and it appears in the literature, and we prove that this space has the fixed point property for non-expansive mappings.

1. Introduction

Let K be a weakly compact, convex subset of a Banach space X. A mapping $T : K \to K$ is called *non-expansive* if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in K$. In the case where every non-expansive map $T : K \to K$ has a fixed point, we say that K has the *fixed point property*. The space X is said to have the fixed point property if every weakly compact, convex subset of X has the fixed point property.

A lot of Banach spaces are known to enjoy the aforementioned property. The earlier results show that uniformly convex spaces have the fixed point property (see [3]) and this is also true for the wider class of spaces with normal structure (see [7]). The classical Banach spaces ℓ_p, L_p with $1 < p < \infty$ are uniformly convex and hence they have the fixed point property. On the contrary, the space L_1 fails this property (see [1]).

The proofs of many positive results depend on the notion of minimal invariant sets. Suppose that K is a weakly compact, convex set, $T : K \to K$ is a non-expansive mapping and C is a nonempty, weakly compact, convex subset of K such that $T(C) \subseteq C$. The set C is called *minimal* for T if there is no strictly smaller weakly compact, convex subset of C which is invariant under T. A straightforward application of Zorn's lemma implies that K always contains minimal invariant subsets. So, a standard approach in proving fixed points theorems is to first assume that K itself is minimal for T and then use the geometrical properties of the space to show that K must be a singleton. Therefore, T has a fixed point.

Although a non-expansive map $T : K \to K$ does not have to have fixed points, it is well-known that T always has an *approximate fixed point sequence*. This means that there is a sequence (x_n) in K such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. For such sequences, the following result holds (see [6]).

Theorem 1.1. *Let K be a weakly compact, convex set in a Banach space, $T : K \to K$ a non-expansive map, such that K is T-minimal, and let (x_n) be any approximate fixed point sequence. Then, for all $x \in K$,*

$$\lim_{n\to\infty} \|x - x_n\| = diam(K).$$

2010 *Mathematics Subject Classification.* Primary 47H10, 47H09, 46B25.

Key words and phrases. Non-expansive mappings, fixed point property, Banach spaces isomorphic to c_0.

Although from the beginning of the theory it became clear that the classical spaces ℓ_p, L_p, $1 < p < \infty$ have the fixed point property, the case of c_0 remained unsolved for some period of time. The geometrical properties of this space are not very nice, in the sense that c_0 does not possess normal structure. However, it was finally proved that the geometry of c_0 is still good enough and and it does not allow the existence of minimal sets with positive diameter, that is c_0 has the fixed point property. This was done by B. Maurey [8] (see also [4]) who also proved that every reflexive subspace of L_1 has the fixed point property.

Theorem 1.2. *The space c_0 has the fixed point property.*

The proof of Theorem 1.2 is based on the fact that the set of approximate fixed point sequences is convex in a natural sense. More precisely, we have the following ([8], [4]).

Theorem 1.3. *Let K be a weakly compact, convex subset of a Banach space which is minimal for a non-expansive map $T : K \to K$. Let (x_n) and (y_n) be approximate fixed point sequences for T such that $\lim_{n\to\infty} \|x_n - y_n\|$ exists. Then there is an approximate fixed point sequence (z_n) in K such that*

$$\lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|y_n - z_n\| = \frac{1}{2} \lim_{n\to\infty} \|x_n - y_n\|.$$

In the present paper, we define a Banach space X isomorphic to c_0 and we prove that this space has the fixed point property. Our interest on this space derives from several reasons. Firstly, the space X comes from c_0 in a natural way. In fact, the Schauder basis of X is equivalent to the summing basis of c_0. Secondly, the space X is close to c_0 in the sense that the Banach-Mazur distance between the two spaces is equal to 2. It is worth mentioning that from the proof of Theorem 1.2 we can conclude that whenever Y is a Banach space isomorphic to c_0 and the Banach-Mazur distance between Y and c_0 is strictly less than 2, then Y has the fixed point property. In our case, the Banach-Mazur distance is equal to 2, that is the space X lies on the boundary of what is already known. This fact should also be compared with the following question in metric fixed point theory: Find a nontrivial class of Banach spaces invariant under isomorphism such that each member of the class has the fixed point property (a trivial example is the class of spaces isomorphic to ℓ_1). We shall see that even for spaces close to c_0, such as the space X, the situation is quite complicated and this points out the difficulty of the aforementioned question. Finally, the space X has been used in several places in the study of the geometry of Banach spaces (for instance see [5], [2]). More precisely, the well-known Hagler Tree space (HT) [5] contains a plethora of subspaces isomorphic to X. Nevertheless, we do not know if HT has the fixed point property.

2. Definition and basic properties

We consider the vector space c_{00} of all real-valued finitely supported sequences. We let $(e_n)_{n\in\mathbb{N}}$ stand for the usual unit vector basis of c_{00}, that is $e_n(i) = 1$ if $i = n$ and $e_n(i) = 0$ if $i \neq n$. If $S \subset \mathbb{N}$ is any *interval* of integers and $x = (x_i) \in c_{00}$ then we set $S^*(x) = \sum_{i\in S} x_i$. We now define the norm of x as follows

$$\|x\| = \sup|S^*(x)|$$

where the supremum is taken over all finite intervals $S \subset \mathbb{N}$. The space X is the completion of the normed space we have just defined.

It is easily verified that the sequence (e_n) is a normalized monotone Schauder basis for the space X. In the following, $(e_n^*)_{n\in\mathbb{N}}$ denotes the sequence of the biorthogonal functionals and $(P_n)_{n\in\mathbb{N}}$ denotes the sequence of the natural projections associated to the basis (e_n). That is, for any $x=\sum_{i=1}^\infty x_ie_i\in X$ we have $e_n^*(x)=x_n$ and $P_n(x)=\sum_{i=1}^n x_ie_i$.

Furthermore, if $S\subset\mathbb{N}$ is any interval of integers (not necessarily finite), we define the functional $S^*:X\to\mathbb{R}$ by $S^*(x)=S^*(\sum_{i=1}^\infty x_ie_i)=\sum_{i\in S}x_i$. It is easy to see that S^* is a bounded linear functional with $\|S^*\|=1$. In the special case where $S=\mathbb{N}$, the corresponding functional is denoted by B^* (instead of the confusing $\mathbb{N}^*$). Therefore, $B^*(x)=\sum_{i=1}^\infty x_i$ for any $x=\sum_{i=1}^\infty x_ie_i\in X$.

The following proposition provides some useful properties of the space X and demonstrates the relation between X and c_0. We remind that for any pair E,F of isomorphic normed spaces, the Banach-Mazur distance between E and F is defined as follows

$$d(E,F)=\inf\{\|T\|\cdot\|T^{-1}\| \mid T:E\to F \text{ is an isomorphism from } E \text{ onto } F\}.$$

Proposition 2.1. *The following hold:*

1. *The space X is isomorphic to c_0 and in particular the basis of X is equivalent to the summing basis of c_0.*
2. *The subspace of X^* generated by the sequence of the biorthogonal functionals has codimension 1. More precisely, $X^*=\overline{span}\{e_n^*\}_{n\in\mathbb{N}}\oplus\langle B^*\rangle$.*
3. *The Banach-Mazur distance $d(X,c_0)$ between X and c_0 is equal to 2.*

Proof. We define the linear operator

$$\begin{aligned}\Phi &: X\to c_0\\ x=(x_i)&\mapsto\Big(\sum_{i=1}^\infty x_i,\sum_{i=2}^\infty x_i,\dots\Big).\end{aligned}$$

It is easily verified that Φ is an isomorphism from X onto c_0 with $\|\Phi\|=1$, $\|\Phi^{-1}\|=2$ and Φ maps the basis of X to the summing basis of c_0. This proves the first assertion. The second assertion is an immediate consequence of the relation between X and c_0 established above.

It remains to show that the Banach-Mazur distance $d=d(X,c_0)$ is equal to 2. Firstly, we observe that the isomorphism Φ defined above implies that $d\le 2$. In order to prove the reverse inequality we fix a real number $\epsilon>0$. Then there exists an isomorphism $T:X\to c_0$ from X onto c_0 such that $\|x\|\le\|Tx\|_{c_0}\le(d+\epsilon)\|x\|$ for any $x\in X$. We now consider the normalized sequence (x_n) in X where $x_n=(x_n(i))_{i\in\mathbb{N}}$ is defined by

$$x_n(2n-1)=-1,\ x_n(2n)=1,\ x_n(i)=0 \text{ otherwise}.$$

The description of X^* given by the second assertion implies that any bounded sequence $(t_n)_{n\in\mathbb{N}}$ of elements of X converges weakly to 0 if and only if $e_m^*(t_n)\to 0$ for every $m\in\mathbb{N}$ and $B^*(t_n)\to 0$. It follows that the sequence $(x_n)_{n\in\mathbb{N}}$ defined above is weakly null. Now we set $y_n=T(x_n)$ for any $n\in\mathbb{N}$ and we have $1\le\|y_n\|_{c_0}\le d+\epsilon$ and $(y_n)_{n\in\mathbb{N}}$ converges weakly to 0. Therefore, we find $k_1\in\mathbb{N}$ such that the vectors y_1 and y_{k_1} have essentially disjoint supports. More precisely, since $y_1\in c_0$, there exists $N_1\in\mathbb{N}$ such that $|y_1(i)|<\epsilon$ for any $i>N_1$. Since $y_n\to 0$ weakly, we find k_1 so that $|y_{k_1}(i)|<\epsilon$ for any $i\le N_1$. It follows that

$\|y_1 - y_{k_1}\|_{c_0} \leq \max\{\|y_1\|_{c_0}, \|y_{k_1}\|_{c_0}\} + \epsilon \leq d + 2\epsilon$. On the other hand, $\|x_1 - x_{k_1}\| = 2$. Therefore,

$$2 = \|x_1 - x_{k_1}\| \leq \|y_1 - y_{k_1}\|_{c_0} \leq d + 2\epsilon.$$

If ϵ tends to 0, we obtain $2 \leq d$ as we desire. □

3. The fixed point property

This section is entirely devoted to the proof of the fixed point property for the space X. First we need to establish some notation. If $S, S' \subset \mathbb{N}$ are intervals we write $S < S'$ to mean that $\max S < \min S'$. Moreover, if $k \in \mathbb{N}$, we write $k < S$ (resp., $S < k$) to mean $k < \min S$ (resp., $\max S < k$). Finally, for any $x = (x_i) \in X$, $\mathrm{supp}(x) = \{i \in \mathbb{N} \mid x_i \neq 0\}$ denotes the support of x.

Theorem 3.1. *The space X has the fixed point property.*

Proof. We follow the standard approach. We assume that K is a weakly compact, convex subset of X which is minimal for a non-expansive map $T : K \to K$. Using the geometry of the space X, we have to show that K is a singleton, that is $diam(K) = 0$. Let us suppose that $diam(K) > 0$ and now we have to reach a contradiction. Without loss of generality we may assume that $diam(K) = 1$.

Let $(x_n)_{n\in\mathbb{N}}$ be an approximate fixed point sequence for the map T in the set K. By passing to a subsequence and then using some translation, we may assume that $0 \in K$ and (x_n) converges weakly to 0. Theorem 1.1 implies that $\lim_n \|x_n\| = diam(K) = 1$.

We next find a subsequence (x_{q_n}) of (x_n) and integers $l_0 = 0 < l_1 < l_2 < \ldots$ such that for every $n \in \mathbb{N}$, $\|P_{l_{n-1}}(x_{q_n})\| < 1/n$ and $\|x_{q_n} - P_{l_n}(x_{q_n})\| < 1/n$. The desired sequences (x_{q_n}) and (l_n) are constructed inductively. We start with $x_{q_1} = x_1$ and $l_0 = 0$. Suppose that $q_1 < q_2 < \ldots < q_n$ and $l_0 < l_1 < \ldots < l_{n-1}$ have been defined. Then there exists $l_n > l_{n-1}$ such that $\|x_{q_n} - P_{l_n}(x_{q_n})\| < 1/n$. Since (x_n) is weakly null, it follows that $P_m(x_n) \to 0$ for every $m \in \mathbb{N}$. Therefore, there exists $q_{n+1} > q_n$ such that $\|P_{l_n}(x_{q_{n+1}})\| < \frac{1}{n+1}$. The construction of (x_{q_n}) and (l_n) is complete.

Consequently, passing to a subsequence, we may assume that for the original sequence (x_n) there are integers $l_0 = 0 < l_1 < l_2 < \ldots$ such that for every $n \in \mathbb{N}$,

$$\|P_{l_{n-1}}(x_n)\| < \frac{1}{n} \text{ and } \|x_n - P_{l_n}(x_n)\| < \frac{1}{n}.$$

As a matter of fact, we can go one step further and suppose that for any $n \in \mathbb{N}$, $P_{l_{n-1}}(x_n) = 0$ and $x_n - P_{l_n}(x_n) = 0$. Therefore, $\mathrm{supp}(x_n) \subset (l_{n-1}, l_n]$, that is (x_n) is a block basis of (e_n). If we did not adopt this assumption, then in each inequality written below we would have to add a term equal to $O(\frac{1}{n})$, which simply would change nothing.

We next consider the subsequences $(z_n) = (x_{2n-1})$ and $(y_n) = (x_{2n})$ and we also set $l_{2n-1} = k_n$ and $l_{2n} = m_n$ for every $n \in \mathbb{N}$. The properties of the sequence (x_n) imply that the following hold.

(1) (z_n) and (y_n) are approximate fixed point sequences for the map T. Therefore, by Theorem 1.1, $\lim \|z_n\| = \lim \|y_n\| = 1$.
(2) (z_n) and (y_n) converge weakly to 0.
(3) $\mathrm{supp}(z_n) \subset (m_{n-1}, k_n]$ and $\mathrm{supp}(y_n) \subset (k_n, m_n]$ for every $n \in \mathbb{N}$.
(4) $\lim \|z_n - y_n\| = 1$.

In order to justify the fourth conclusion, we first observe that $\lim \|z_n - y_n\| \leq diam(K) = 1$. On the other hand, by the definition of the norm of the space X, for every $n \in \mathbb{N}$ there exists a finite interval $E_n \subset \mathbb{N}$ such that $\|z_n\| = |E_n^*(z_n)|$. Clearly we may assume that $E_n \subset (m_{n-1}, k_n]$. Then $\|z_n - y_n\| \geq |E_n^*(z_n - y_n)| = \|z_n\|$ and therefore $\lim \|z_n - y_n\| \geq \lim \|z_n\| = 1$.

We are ready now to apply Maurey's theorem (Theorem 1.3). To this end, we fix a positive integer $N \in \mathbb{N}$, which will be chosen properly at the end of the proof, and we set $\epsilon = 2^{-N}$. After N iterated applications of Theorem 1.3 we find a sequence (v_n) in the set K such that: (v_n) is an approximate fixed point sequence for the map T (which implies that $\lim \|v_n\| = 1$) and further $\lim \|v_n - z_n\| = \epsilon$ and $\lim \|v_n - y_n\| = 1 - \epsilon$. Therefore, for all sufficiently large $n \in \mathbb{N}$ the following hold:

(1) $\|v_n\| > 1 - \frac{\epsilon}{2}$;
(2) $\|v_n - z_n\| < 3\epsilon/2$ and $\|v_n - y_n\| < 1 - \frac{\epsilon}{2}$;
(3) $|B^*(z_n)| < \epsilon/2$ (since (z_n) is weakly null).

We also set $S_n = (m_{n-1}, k_n]$ so that we have $S_1 < S_2 < \ldots$. Concerning the sequence (v_n) in the set K and the sequence of intervals (S_n) we prove the following two claims.

Claim 1. For all sufficiently large n, the support of v_n is essentially contained in the interval S_n, in the sense that if S is any interval with $S \cap S_n = \emptyset$ then $|S^*(v_n)| < 3\epsilon/2$.

Indeed, we know that $\text{supp}(z_n) \subset (m_{n-1}, k_n] = S_n$. Therefore, if S is any interval with $S \cap S_n = \emptyset$ then $S^*(z_n) = 0$ and hence

$$|S^*(v_n)| = |S^*(v_n - z_n)| \leq \|v_n - z_n\| < \frac{3\epsilon}{2}.$$

Claim 2. For all sufficiently large n, there exist intervals $L_n < R_n$ such that $S_n = L_n \cup R_n$ and $L_n^*(v_n) < -1 + 7\epsilon$, $R_n^*(v_n) > 1 - 2\epsilon$.

We fix a sufficiently large positive integer n. Since $\|v_n\| > 1 - \frac{\epsilon}{2}$, it follows that there exists a finite interval $F_n \subset \mathbb{N}$ such that $|F_n^*(v_n)| > 1 - \frac{\epsilon}{2}$. If $k_n < F_n$, we know by the previous claim that $|F_n^*(v_n)| < 3\epsilon/2$, which is a contradiction. Moreover, if we assume that $F_n \leq k_n$ then $F_n \cap (k_n, m_n] = \emptyset$ and the choice of (y_n) implies $F_n^*(y_n) = 0$. Thus,

$$|F_n^*(v_n)| = |F_n^*(v_n - y_n)| \leq \|v_n - y_n\| < 1 - \frac{\epsilon}{2},$$

which is also a contradiction. By this discussion it is clear that $\min F_n \leq k_n < \max F_n$. Now we set $R_n = F_n \cap [1, k_n]$ and we estimate

$$1 - \frac{\epsilon}{2} < |F_n^*(v_n)| \leq |R_n^*(v_n)| + |(F_n \setminus R_n)^*(v_n)| < |R_n^*(v_n)| + \frac{3\epsilon}{2},$$

where the last inequality follows by Claim 1. Therefore, $|R_n^*(v_n)| > 1 - 2\epsilon$. Passing to a subsequence, we may assume that either $R_n^*(v_n) > 1 - 2\epsilon$ for all sufficiently large n or $R_n^*(v_n) < -1 + 2\epsilon$ for all sufficiently large n. We suppose that the first possibility happens, as the second one is treated similarly (interchanging the roles of L_n and R_n). Consequently, for the interval R_n we have $\max R_n = k_n$ and $R_n^*(v_n) > 1 - 2\epsilon$.

On the other hand, we observe that

$$|B^*(v_n)| \leq |B^*(v_n - z_n)| + |B^*(z_n)| \leq \|v_n - z_n\| + \frac{\epsilon}{2} < 2\epsilon.$$

We note that the sequence (v_n) is not necessarily weakly null. However, v_n is close to z_n and hence $|B^*(v_n)|$ is very small. We next set $G_n = [1, \min R_n)$ (possibly empty) and $W_n = (k_n, +\infty)$. Then,

$$\begin{aligned} 2\epsilon > |B^*(v_n)| &= |G_n^*(v_n) + R_n^*(v_n) + W_n^*(v_n)| \\ &\geq R_n^*(v_n) - |G_n^*(v_n)| - |W_n^*(v_n)| \\ &> 1 - 2\epsilon - |G_n^*(v_n)| - \frac{3\epsilon}{2}. \end{aligned}$$

Therefore G_n is non-empty and $|G_n^*(v_n)| > 1 - \frac{11\epsilon}{2}$. However, if $G_n^*(v_n) > 1 - \frac{11\epsilon}{2}$, then it would follow

$$|B^*(v_n)| \geq R_n^*(v_n) + G_n^*(v_n) - |W_n^*(v_n)| \geq 2 - 9\epsilon,$$

which is a contradiction. Hence, $G_n^*(v_n) < -1 + \frac{11\epsilon}{2}$. Further, we observe that we can not have $G_n < S_n$, since in this case it would follow $|G_n^*(v_n)| < \frac{3\epsilon}{2}$. Consequently, $\max G_n > m_{n-1}$ which clearly implies $\min R_n > m_{n-1} + 1$. Finally, we set $L_n = G_n \cap (m_{n-1}, k_n]$ and we estimate

$$-1 + \frac{11\epsilon}{2} > G_n^*(v_n) = L_n^*(v_n) + (G_n \setminus L_n)^*(v_n) \geq L_n^*(v_n) - \frac{3\epsilon}{2}.$$

We deduce that $L_n^*(v_n) < -1 + 7\epsilon$. Therefore, the intervals $L_n < R_n$ satisfy the following: $S_n = L_n \cup R_n$, $R_n^*(v_n) > 1 - 2\epsilon$ and $L_n^*(v_n) < -1 + 7\epsilon$. The proof of the claim is now complete.

Using the construction and the properties of the sequences (v_n) and (S_n), we can reach the final contradiction and finish the proof of the theorem. Indeed, we fix a sufficiently large $n \in \mathbb{N}$ and we consider the intervals $D = (k_n, m_n]$ and $S = R_n \cup D \cup L_{n+1}$. Then, using Claim 1 and Claim 2 we have

$$\begin{aligned} S^*(v_n) &= R_n^*(v_n) + (D \cup L_{n+1})^*(v_n) > 1 - 2\epsilon - \frac{3\epsilon}{2} = 1 - \frac{7\epsilon}{2} \\ S^*(v_{n+1}) &= (R_n \cup D)^*(v_{n+1}) + L_{n+1}^*(v_{n+1}) < \frac{3\epsilon}{2} - 1 + 7\epsilon = -1 + \frac{17\epsilon}{2}. \end{aligned}$$

Therefore,

$$\|v_n - v_{n+1}\| \geq |S^*(v_n - v_{n+1})| = |S^*(v_n) - S^*(v_{n+1})| \geq 2 - 12\epsilon.$$

If ϵ has been chosen small enough, then we have a contradiction, since $\|v_n - v_{n+1}\| \leq diam(K) = 1$. □

References

1. D. Alspach, *A fixed point free nonexpansive map*, Proc. Amer. Math. Soc. **82** (1981), 423-424.
2. S. A. Argyros, I. Deliyanni and A. G. Tolias, *Hereditarily indecomposable Banach Algebras of diagonal operators*, Israel J. Math. **181** (2011), 65-110.
3. F. E. Browder, *Nonexpansive nonlinear operators in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041-1044.
4. J. Elton, P. Lin, E. Odell and S. Szarek, *Remarks on the fixed point problem for nonexpansive maps*, Contemporary Math. **18** (1983), 87-119.
5. J. Hagler, *A counterexample to several questions about Banach spaces*, Studia Math. **60** (1977), 289-308.
6. L. A. Karlovitz, *Existence of fixed points for nonexpansive mappings in a space without normal structure*, Pacific J. Math. **66** (1976), 153-159.
7. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004-1006.

8. B. Maurey, *Points fixes des contractions sur un convexe forme de* L^1, Seminaire d' Analyse Fonctionelle 80-81, Ecole Polytechnique, Palaiseau.

Department of Mathematics, University of Athens, 15784, Athens, Greece
E-mail address: k-poulios@math.uoa.gr